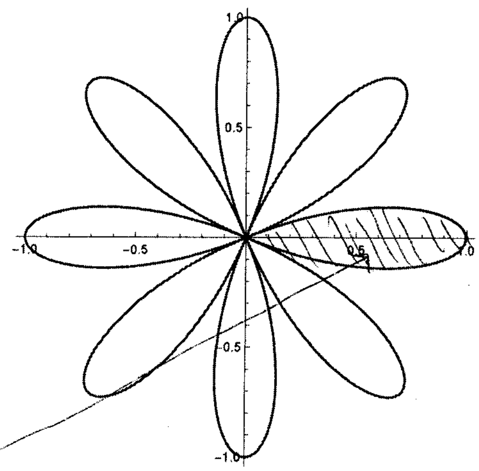
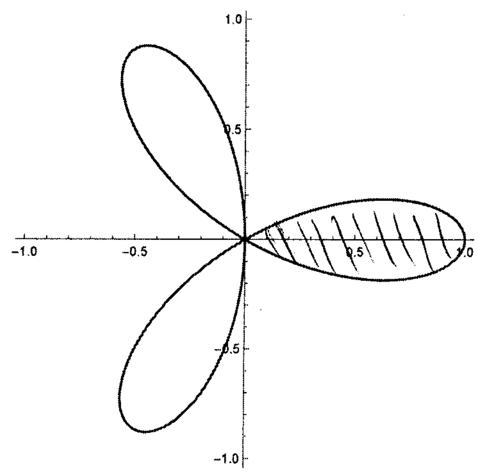
Polar Remix (feat. Shorlando Shbloom)
MATH 116-022 (Lutz)

1. Write separate integrals giving the areas of the regions bounded by the following rose curves. Then compute the perimeter of one lobe of each.



(a) $r = \cos(4\theta)$

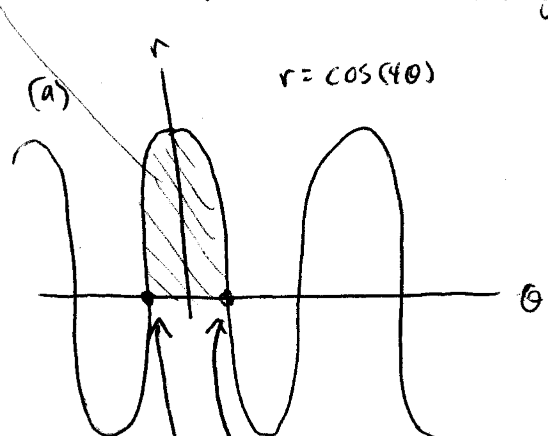


(b) $r = \cos(3\theta)$

Let's find the angles corresponding to one petal of each.

To do this, make a Cartesian graph of r as a function of θ.

(a)



A petal is traced out between each consecutive value of θ where $r = 0$.

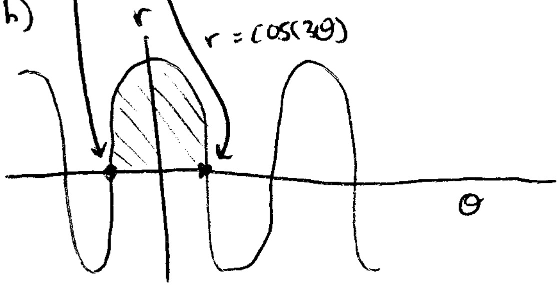
Area of one petal is $\frac{1}{2}\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} (\cos(4\theta))^2 \, d\theta$

So, since there are 8 petals:

$$\boxed{\text{Total area: } \frac{8}{2}\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} (\cos(4\theta))^2 \, d\theta.}$$

$-\frac{\pi}{6}$ $\frac{\pi}{6}$ $-\frac{\pi}{8}$ $\frac{\pi}{8}$

(b)



$$\boxed{\text{Total area: } \frac{3}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (\cos(3\theta))^2 \, d\theta}$$

For the perimeter of a single petal we use the formula for polar arc length $\int_{\theta_a}^{\theta_b} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

a) $f(\theta) = \cos(4\theta)$

$f'(\theta) = -4\sin(4\theta)$

$$\boxed{\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \sqrt{\cos^2(4\theta) + 16\sin^2(4\theta)}\, d\theta}$$

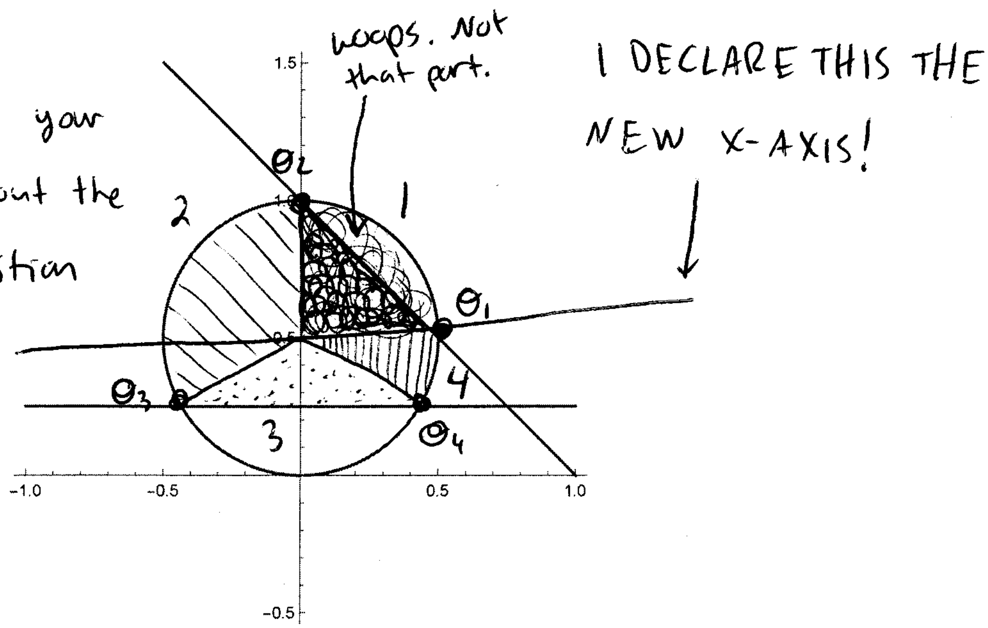
b) $f(\theta) = \cos(3\theta)$

$f'(\theta) = -3\sin(3\theta)$

$$\boxed{\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sqrt{\cos^2(3\theta) + 9\sin^2(3\theta)}\, d\theta.}$$

2. Write an integral giving the area of the region bounded by the circle $r = \sin(\theta)$, the line $y = 1 - x$, and the line $y = \frac{1}{4}$, pictured below. (Hint: The going is easier if you translate down by $\frac{1}{2}$.)

Polar works best when your curves are symmetric about the origin. Hence the suggestion about translating down.



The equation of the circle is now simply $r = \frac{1}{2}$.

In truth, I wouldn't use integrals to compute the area of this region; it's just a couple sectors and a couple triangles. But I'm a good citizen and I do what I'm told.

There are 4 regions. I write 4 integrals. Regions 2 and 4 are the easiest.

Let's first figure out the angles.

$\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$ using my eyes. For the other two I first use the triangle

This angle is ψ

and $\cos\psi = \frac{1/4}{1/2} = \frac{1}{2}$, so $\psi = \frac{\pi}{3} \Longrightarrow \theta_3 = \frac{3\pi}{2} - \psi$

$= \frac{9\pi}{6} - \frac{2\pi}{6} = \frac{7\pi}{6}$.

and $\theta_4 = \theta_3 + \frac{2\pi}{3} = \frac{11\pi}{6}$.

$\theta_3 = \frac{7\pi}{6} \quad \theta_4 = \frac{11\pi}{6}$

Next, I write the lines in polar.

$$y = 1 - x \xrightarrow[\text{down}]{\text{shift}} y = 1 - x - \frac{1}{2} = \frac{1}{2} - x$$

$$y = \frac{1}{4} \xrightarrow[\text{down}]{\text{shift}} y = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$

Now convert. 1. $r\sin\theta = y = \frac{1}{2} - x = \frac{1}{2} - r\cos\theta$

$$r(\sin\theta + \cos\theta) = \frac{1}{2}$$

$$r = \frac{1}{2(\sin\theta + \cos\theta)} \quad \checkmark.$$

2. $r\sin\theta = y = \frac{1}{4}$

$$r = \frac{1}{4\sin\theta} \quad \checkmark.$$

Finally, the integrals.

1. 2. 3.

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1}{2(\sin\theta + \cos\theta)}\right)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{7\pi}{6}} \left(\frac{1}{2}\right)^2 d\theta + \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \left(\frac{1}{4\sin\theta}\right)^2 d\theta$$

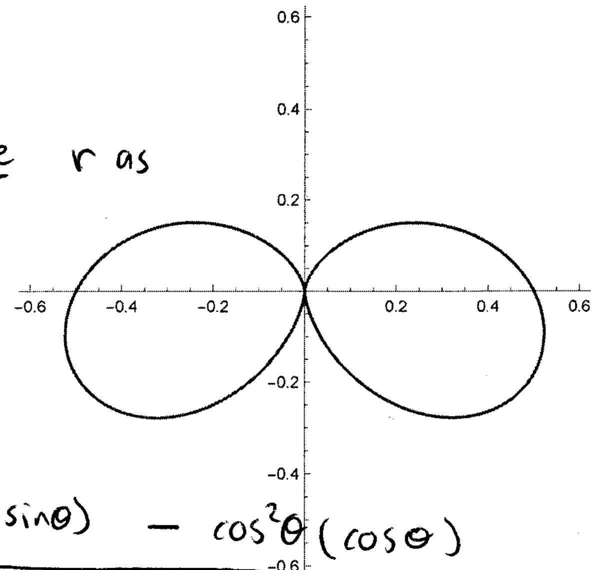
4.

$$+ \frac{1}{2}\int_{\frac{11\pi}{6}}^{2\pi} \left(\frac{1}{2}\right)^2 d\theta.$$

3. Shorlando Shbloom is customizing his new sunglasses online. He's decided on lenses outlined by the polar curve $r = \cos^2(\theta)/(2+\sin(\theta))$, plotted below. Answer the following questions about his dope new shades.

(a) For which $\theta \in [0, 2\pi)$ is the curve farthest from the origin?

(b) What is the perimeter of one lens?



a) we want to maximize r as a function of θ.

Use calculus.

$$\frac{dr}{d\theta} = \frac{(2+\sin\theta)\,2\cos\theta(-\sin\theta) - \cos^2\theta(\cos\theta)}{(2+\sin\theta)^2} = 0$$

$$-4\sin\theta\cos\theta - 2\sin^2\theta\cos\theta - \cos^3\theta = 0$$

$$\cos\theta\left(2\sin^2\theta + 4\sin\theta + \cos^2\theta\right) = 0$$

$\hookrightarrow$ $\cos\theta = 0 \Longrightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$; look at the picture, these aren't what we want.

or

$$2\sin^2\theta + 4\sin\theta + (1-\sin^2\theta) = 0$$

$\sin^2\theta + 4\sin\theta + 1 = 0$ This is a quadratic in $\sin\theta$.

$$\sin\theta = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

$\hookrightarrow$ $\theta = \arcsin\left(-2 \pm \sqrt{3}\right)$ ... that looks right.

b) Look for two consecutive angles where $r = 0$.

$$r = \frac{\cos^2\theta}{2+\sin\theta} = 0 \implies \cos^2\theta = 0 \implies \cos\theta = 0$$

$\implies$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are lense must be traced.

$$f(\theta) = \frac{\cos^2\theta}{2+\sin\theta}$$

$$f'(\theta) = \frac{-4\sin\theta\cos\theta - 2\sin^2\theta\cos\theta - \cos^3\theta}{(2+\sin\theta)^2}$$

perimeter of are lense :

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\left(\frac{\cos^2\theta}{(2+\sin\theta)}\right)^2 + \left(\frac{-4\sin\theta\cos\theta - 2\sin^2\theta\cos\theta - \cos^3\theta}{(2+\sin\theta)^2}\right)^2}\, d\theta.$$

4. Shorlando has been turned into a horrible bean creature whose shape is the polar curve $r = \sin(2\theta) + \cos(\theta)$, pictured below. Answer the following questions about his hot new look.

   (a) What is the lowest point on the curve?

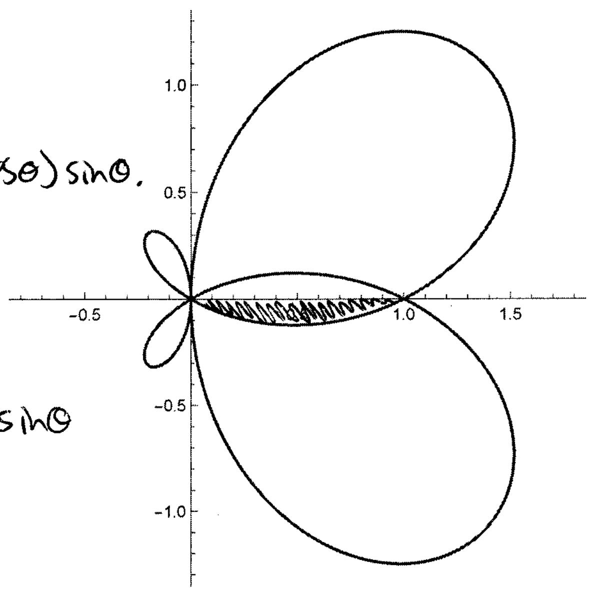
   (b) What are the rightmost points on the curve?

   (c) What is the tangent line at the points you found in (b)?

   (d) Shorlando's new liver is the football-shaped region between his horrible bean lobes. What is the area of his new liver?

a) minimize $y = r\sin\theta$

$= (\sin 2\theta + \cos\theta)\sin\theta.$



$$\frac{dy}{d\theta} = (\sin 2\theta + \cos\theta)\cos\theta + (2\cos 2\theta - \sin\theta)\sin\theta = 0$$

$$\sin 2\theta \cos\theta + \cos^2\theta + 2\cos 2\theta \sin\theta - \sin^2\theta = 0$$

$\cos^2\theta - \sin^2\theta = \cos(2\theta)$

$\sin(2\theta)\cos\theta + \cos(2\theta) + 2\cos(2\theta)\sin\theta = 0$ ... Just use your calculator to find the (many) solutions. Check which one gives the lowest value for y.

The calculus is actually pointless here, since you can't solve easily by hand. I would just graph y as a function of θ and locate the lowest point visually.

The lowest point occurs at $\theta \approx 2.238$

$y = -1.25.$

b) just plot $x = r\cos\theta = (\sin(2\theta) + \cos(\theta))\cos\theta$ and locate

the ~~...~~ largest values.

$x \approx 1.516$, $\theta_1 = 0.449$

$\theta_2 = 2.692$

c) since these are "rightmost points" their tangent line is vertical.

$$\boxed{x = 1.516}$$

d) The simplest way to do this is to double the area of half the region.

Using a cartesian graph of $r = \sin(2\theta) + \cos(\theta)$ I see that the shaded bottom half

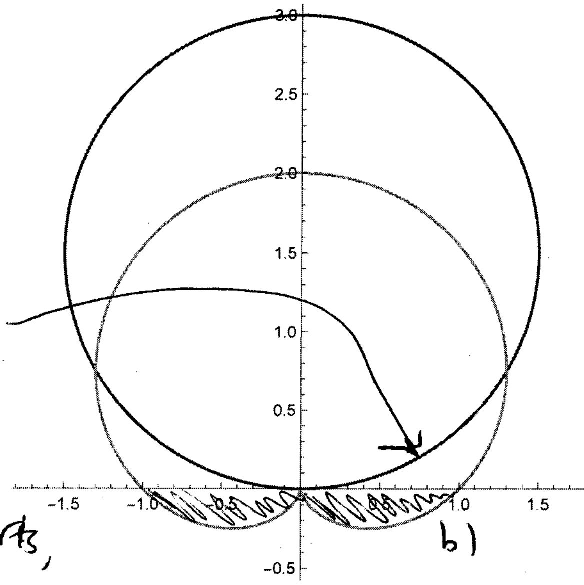
starts at $\theta = -\frac{\pi}{6}$ and ends at $\theta = 0$.

$$\boxed{2 \cdot \frac{1}{2} \int_{-\frac{\pi}{6}}^{0} (\sin(2\theta) + \cos(\theta))^2 \, d\theta}$$

5. Shorlando's new bean rump is stuck.

(a) He implores you to compute the total area of stuck rump, which is outside the circle $r = 3\sin(\theta)$ and inside the limaçon $r = 1 + \sin(\theta)$, as pictured below.

(b) After wiggling his bean body spiritedly, he has freed some precious rump. Compute the new area of stuck rump, which is now the region inside the limaçon and below the x-axis.



a) & b) We do both parts in one. I'm only going to look at half the butt, and double my answer.

The region splits into two parts, the shaded and unshaded parts.

Shaded bounds: $\frac{3\pi}{2}$ to 2π

Unshaded bounds: 0 to $\frac{\pi}{6}$

Point of intersection.

$3\sin(\theta) = 1 + \sin\theta$

$\sin\theta = \frac{1}{2}$ $\quad \theta = \frac{\pi}{6}.$

Total shaded area.

Total unshaded area.

$$\underbrace{2\,\frac{1}{2}\int_{\frac{3\pi}{2}}^{2\pi}(1+\sin\theta)^2\,d\theta}_{\text{answer to (b)}} + 2\left(\frac{1}{2}\int_{0}^{\frac{\pi}{6}}(1+\sin\theta)^2\,d\theta - \frac{1}{2}\int_{0}^{\frac{\pi}{6}}(3\sin\theta)^2\,d\theta\right)$$

6. In the transformation, Shorlando's heart turned into the Arby's logo. The transformation is captured by the family of polar curves $r = \cos(5t) + \mu\cos(t)$, where μ varies in some interval $[a, b]$. Answer the following questions to help out poor Shorlando.

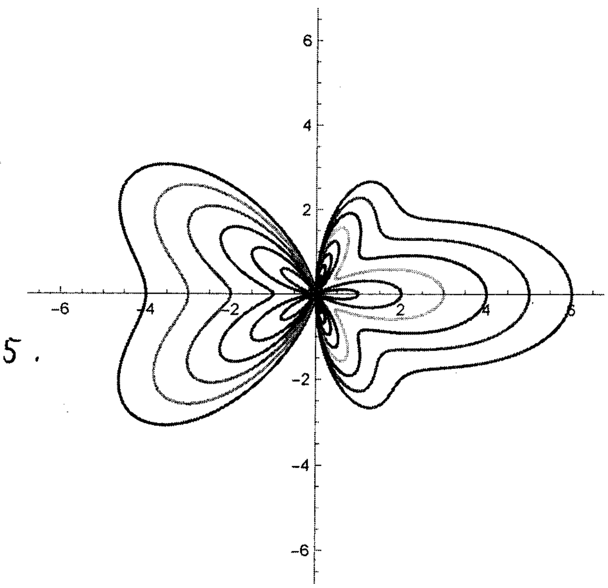
   (a) Assuming the extreme cases (i.e., $\mu = a$ and $\mu = b$) are among those pictured below, what are the values of a and b?

   (b) For which value of μ is the tangent line to the curve at $\theta = \frac{\pi}{20}$ horizontal? Give your answer in exact form.

a) evaluate @ $\theta = 0$

$r(0) = 1 + \mu$.

Since one curve reaches out to $r = 6$, b must be $6 - 1 = 5$.



The lowest it gets is -4

So $1 + a = -4$

$a = -5$ $\quad \therefore\ a = -5$ and $b = 5$

you should graph to double check.

b) Tangent line horizontal when $\frac{dy}{d\theta} = 0$. $\quad y = (\cos(5\theta) + \mu\cos(\theta))\sin\theta$

$$\frac{dy}{d\theta} = (\cos(5\theta) + \mu\cos(\theta))\cos\theta + (-5\sin(5\theta) - \mu\sin(\theta))\sin\theta = 0.$$

Now evaluate @ $\theta = \frac{\pi}{20}$

Solve for μ.

$$\left(\cos\left(\tfrac{\pi}{4}\right) + \mu\cos\left(\tfrac{\pi}{20}\right)\right)\cos\left(\tfrac{\pi}{20}\right) - \left(5\sin\left(\tfrac{\pi}{4}\right) + \mu\sin\left(\tfrac{\pi}{20}\right)\right)\sin\left(\tfrac{\pi}{20}\right) = 0$$

$$\boxed{\mu = \frac{5\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{20}\right) - \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{20}\right)}{\cos\left(\frac{\pi}{20}\right)^2 - \sin\left(\frac{\pi}{20}\right)^2}}$$